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AN ANALOG TO DE MOIVRE'S THEOREM IN A PLANE POINT SYSTEM.

By Prof. E. W. Hyde, Cincinnati, O.

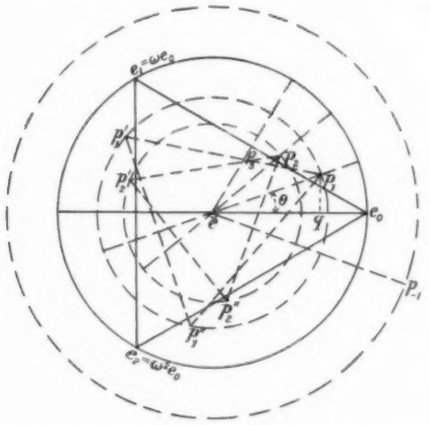
Let e_0, e_1, e_2 be three reference points for plane space, situated at the three vertices of an equilateral triangle, as in the figure; and let ω be an operator which changes e_0 into e_1, e_1 into e_2, and e_2 into e_0, i. e. rotates any point through 120°, about $\bar{e}$, the centroid of the reference triangle. Then we shall have $\omega e_0 = e_1$, $\omega^2 e_0 = \omega e_1 = e_2$, $\omega^3 e_0 = \omega^2 e_1 = \omega e_2 = e_0$, etc., or $\omega^3 = 1$, $\omega^4 = \omega$, $\omega^5 = \omega^2$, $\omega^6 = 1$, etc.

Of course we may, if we choose, regard ω as $\frac{1}{2}(-1 + \sqrt{-3})$, one of the imaginary cube roots of unity, which possesses the above properties, but the work which follows does not depend upon its having this value.

It is proposed to investigate the operator $x_0 + x_1\omega + x_2\omega^2$ which transforms a point in a manner analogous to that in which $x + iy$ does a vector.

We write

$$(x_0 + \omega x_1 + \omega^2 x_2)\, e_0 = x_0 e_0 + x_1 e_1 + x_2 e_2 = p_1 .$$



Now let us see what the effect will be of applying this operator again.

We have

$$\begin{aligned}
&(x_0 + \omega x_1 + \omega^2 x_2)^2\, e_0 \\
&= (x_0 + \omega x_1 + \omega^2 x_2)\, p_1 \\
&= p_2 = (x_0 + \omega x_1 + \omega^2 x_2)(x_0 + \omega x_1 + \omega^2 x_2)\, e_0 \\
&\quad = x_0 (x_0 + \omega x_1 + \omega^2 x_2)\, e_0 + x_1 (\omega x_0 + \omega^2 x_1 + x_2)\, e_0 + x_2 (\omega^2 x_0 + x_1 + \omega x_2) e_0 \\
&\quad = x_0 p_1 + x_1 p_1' + x_2 p_1'' ,
\end{aligned}$$

say, in which p_1' is related to e_1, e_2, e_0 as p_1 is to e_0, e_1, e_2, and p_1'' is related to e_2, e_0, e_1 as p_1 is to e_0, e_1, e_2, i. e. $p_1' = \omega p_1$ and $p_1'' = \omega p_1' = \omega^2 p_1$. The location of p_1, p_1', p_1'' is shown in the figure.

Operating again, we have

$$(x_0 + \omega x_1 + \omega^2 x_2)\, p_2 = p_3 = (x_0 + \omega x_1 + \omega^2 x_2)\,(x_0 + \omega x_1 + \omega^2 x_2)\, p_1$$

$$= x_0\,(x_0 + \omega x_1 + \omega^2 x_2)\, p_1 + x_1\,(\omega x_0 + \omega^2 x_1 + x_2)\, p_1 + x_2\,(\omega^2 x_0 + x_1 + \omega x_2)\, p_1$$

$$= x_0 p_2 + x_1 p_2' + x_2 p_2'' = x_0 p_2 + x_1 \omega p_2 + x_2 \omega^2 p_2\,,$$

so that we have a new set of points p_2, p_2', p_2'', related to p_1, p_1', p_1'', as these are related to e_0, e_1, e_2. This process may evidently be repeated indefinitely.

Now these sets of points lie each on a circle whose center is at e, and the radii of these circles vary in length according to the following law. Let $Tee_0 = a$, and $Tep_1 = na$; then $Tep_2 = n^2 a$, $Tep_k = n^k a$. Again (angle between ee_0 and ep_1) = (angle between ep_1 and ep_2) = (angle between ep_2 and ep_3), etc. Hence the operator $x_0 + \omega x_1 + \omega^2 x_2$ is a function of the angle θ between ee_0 and ep_1, and of the ratio n of the tensors Tep_1 and Tee_0; while $(x_0 + \omega x_1 + \omega^2 x_2)^k$ is a function of $k\theta$ and n^k. Again consider the point p_{-1} for which the angle is $-\theta$, and the distance from e is $\frac{1}{n}\,a$. The operator acting upon this point will evidently move it to e_0, that is $(x_0 + \omega x_1 + \omega^2 x_2)\, p_{-1} = e_0$, whence

$$p_{-1} = (x_0 + \omega x_1 + \omega^2 x_2)^{-1}\, e_0\,.$$

Evidently we shall have in the same way

$$p_{-k} = (x_0 + \omega x_1 + \omega^2 x_2)^{-k}\, e_0$$

in which, as well as in the expression above, k is to be taken as an integer. p_k is taken as a *unit* point, which requires that we have

$$x_0 + x_1 + x_2 = 1\,. \qquad (1)$$

We will now determine n and θ as functions of the x's, and also $x_0 \ldots x_1$ as functions of n and θ.

We have for the length of the line joining two points $p_1 = \Sigma l e$ and $p_2 = \Sigma m e$,*

$$T(p_2 - p_1) = a\sqrt{3[(m_1 - l_1)^2 + (m_2 - l_2)^2 + (m_1 - l_1)\,(m_2 - l_2)}\,,$$

* See Hyde's Directional Calculus, Eq. 93.

which may, by symmetry, be also written

$$T(p_2 - p_1) = a\sqrt{3}\sqrt{(l_0 - m_0)^2 + (l_1 - m_1)^2 + (l_0 - m_0)(l_1 - m_1)}.$$

In this equation put (x_0, x_1, x_2) for (l_0, l_1, l_2), and let $m_0 = m_1 = m_2 = \frac{1}{3}$; then

$$Tep_1 = na = a\sqrt{3}\sqrt{(x_0 - \tfrac{1}{3})^2 + (x_1 - \tfrac{1}{3})^2 + (x_0 - \tfrac{1}{3})(x_1 - \tfrac{1}{3})},$$

or

$$n = \sqrt{3}\sqrt{(x_0 - \tfrac{1}{3})^2 + (x_1 - \tfrac{1}{3})^2 + (x_0 - \tfrac{1}{3})(x_1 - \tfrac{1}{3})}. \qquad (2)$$

Let us write $p_1 - e_0 = \rho$, $e_1 - e_0 = \varepsilon_1$, $e_2 - e_0 = \varepsilon_2$; $\therefore\ T(\varepsilon_1 + \varepsilon_2) = 3a$, and $p_1 - e = \rho - \frac{1}{3}(\varepsilon_1 + \varepsilon_2)$.

$$\operatorname{Cos}\theta = \frac{Teq}{Tep_1} = \frac{a - \rho}{na}\frac{U(\varepsilon_1 + \varepsilon_2)}{} = \frac{3a^2 - \rho}{3na^2}\frac{(\varepsilon_1 + \varepsilon_2)}{}.$$

But

$$(\varepsilon_1 + \varepsilon_2) = 3aU(\varepsilon_2 - \varepsilon_1) = (\varepsilon_2 - \varepsilon_1)\sqrt{3};$$

$$\therefore \qquad \cos\theta = \frac{3a^2 - \rho(\varepsilon_2 - \varepsilon_1)\sqrt{3}}{3na^2} = \frac{3a^2 - e_0 p_1 (e_2 - e_1)\sqrt{3}}{3na^2}$$

$$= \frac{3a^2 - \sqrt{3}(p_1 e_2 e_0 + p_1 e_0 e_1)}{3na^2} = \frac{3a^2 - \sqrt{3}(x_1 + x_2)}{3na^2}.$$

Now we always assume the double area of the reference triangle as unity,

$$\therefore \quad e_0 e_1 e_2 = 1 = a\sqrt{3}\cdot\tfrac{3}{2}a = \tfrac{3}{2}a^2\sqrt{3},$$

whence

$$a^2 = \frac{2}{3\sqrt{3}},$$

therefore

$$\cos\theta = \frac{2 - 3(x_1 + x_2)}{2n}. \qquad (3)$$

From (3) and (1) we have

$$x_1 + x_2 = 1 - x_0 = \tfrac{2}{3}(1 - n\cos\theta),$$

whence

$$x_0 = \tfrac{1}{3}(1 + 2n\cos\theta). \qquad (4)$$

Substituting this value of x_0 in eq. (2) we have

$$n^2 = 3\left[\tfrac{4}{9}n^2\cos^2\theta + (x_1 - \tfrac{1}{3})^2 + \tfrac{2}{3}n\cos\theta(x_1 - \tfrac{1}{3})\right],$$

from which we obtain

$$x_1 = \tfrac{1}{3}(1 - n\cos\theta + n\sqrt{3}\sin\theta) = \tfrac{1}{3}\left[1 - 2n\cos\left(\theta + \frac{\pi}{3}\right)\right], \qquad (5)$$

and by (1) and (4)

$$x_2 = \tfrac{1}{3}(1 - n\cos\theta - n\sqrt{3}\sin\theta) = \tfrac{1}{3}\left[1 - 2n\cos\left(\theta - \frac{\pi}{3}\right)\right]. \qquad (6)$$

We have thus found the x's as functions of n and θ: let us call them K functions; thus

$$\left.\begin{aligned} K_0(n,\theta) &= \tfrac{1}{3}(1 + 2n\cos\theta) \\ K_1(n,\theta) &= \tfrac{1}{3}\left[1 - 2n\cos\left(\theta + \frac{\pi}{3}\right)\right] \\ K_2(n,\theta) &= \tfrac{1}{3}\left[1 - 2n\cos\left(\theta - \frac{\pi}{3}\right)\right] \end{aligned}\right\}, \qquad (7)$$

and we have

$$K_0(n,\theta) + K_1(n,\theta) + K_2(n,\theta) = 1, \qquad (8)$$

an equation analogous to $\cos^2\theta + \sin^2\theta = 1$.

Using K's now for x's we may write

$$(K_0(n,\theta) + \omega K_1(n,\theta) + \omega^2 K_2(n,\theta))^k = K_0(n^k, k\theta) + \omega K_1(n^k, k\theta) + \omega^2 K_2(n^k, k\theta), \qquad (9)$$

which is the analog of De Moivre's theorem. We have shown it to be true for all integral values of k, and will now show it to hold for fractional values. Let us write for brevity

$$F(n,\theta) = K_0(n,\theta) + \omega K_1(n,\theta) + \omega^2 K_2(n,\theta), \qquad (10)$$

then, since, when k is an integer, we have

$$[F(n,\theta)]^k = F(n^k, k\theta),$$

therefore $F(n,\theta)$ is one value of

$$[F(n^k, k\theta)]^{1/k}.$$

But

$$[F(n,\theta)]^{l/k} = \{[F(n,\theta)]^l\}^{1/k} = [F(n^l, l\theta)]^{1/k},$$

of which, as above, one value is

$$F\left[n^{l/k}, \frac{l}{k}\theta\right],$$

and hence the theorem holds for all real values of k.

To find the k values of the kth root we have, as in De Moivre's Theorem,

$$F(n, \theta) = F(n, \theta + 2r\pi),$$

$$\therefore \quad [F(n, \theta)]^{1/k} = [F(n, \theta + 2r\pi)]^{1/k} = F\left[n^{1/k}, \frac{\theta + 2r\pi}{k}\right]. \tag{11}$$

Addition-multiplication Theorem for the K-functions.

Let

$$p_1 = (x_0 + \omega x_1 + \omega^2 x_2)\, e_0, \quad q_1 = (y_0 + \omega y_1 + \omega^2 y_2)\, e_0,$$

and

$$p' = (x_0 + \omega x_1 + \omega^2 x_2)\, q_1 = (x_0 + \omega x_1 + \omega^2 x_2)(y_0 + \omega y_1 + \omega^2 y_2)\, e_0$$
$$= (y_0 + \omega y_1 + \omega^2 y_2)(x_0 + \omega x_1 + \omega^2 x_2)\, e_0 = (y_0 + \omega y_1 + \omega^2 y_2)\, p_1.$$

These equations show that p' is obtained by performing the x operation on q_1, or the y operation on p_1. The result is evidently to turn ee_0 through the angle $\theta_1 + \theta_2$ and to multiply its length by $n_1 n_2$; then expanding the value of p', and putting $K_0(n_1, \theta_1)$ for x_0, $K_0(n_2, \theta_2)$ for y_0, etc., we have

$$p' = [(x_0y_0 + x_1y_2 + x_2y_1) + (x_2y_2 + x_1y_0 + x_0y_1)\,\omega + (x_1y_1 + x_2y_0 + x_0y_2)\,\omega^2]\, e_0$$
$$= [K_0(n_1n_2, \theta_1 + \theta_2) + \omega K_1(n_1n_2, \theta_1 + \theta_2) + \omega^2 K_2(n_1n_2, \theta_1 + \theta_2)]\, e_0.$$

$$\therefore \quad \left.\begin{aligned} K_0(n_1n_2, \theta_1 + \theta_2) &= K_0(n_1, \theta_1)\, K_0(n_2, \theta_2) + K_1(n_1, \theta_1)\, K_2(n_2, \theta_2) \\ &\quad + K_2(n_1, \theta_1)\, K_1(n_2, \theta_2) \\ K_1(n_1n_2, \theta_1 + \theta_2) &= K_2(n_1, \theta_1)\, K_2(n_2, \theta_2) + K_0(n_1, \theta_1)\, K_1(n_2, \theta_2) \\ &\quad + K_1(n_1, \theta_1)\, K_0(n_2, \theta_2) \\ K_2(n_1n_2, \theta_1 + \theta_2) &= K_1(n_1, \theta_1)\, K_1(n_2, \theta_2) + K_2(n_1, \theta_1)\, K_0(n_2, \theta_2) \\ &\quad + K_0(n_1, \theta_1)\, K_2(n_2, \theta_2) \end{aligned}\right\}. \tag{12}$$

If in these equations we make $n_1 = n_2 = n$, and $\theta_1 = \theta_2 = \theta$, we have

$$\left.\begin{aligned} K_0(n^2, 2\theta) &= [K_0(n, \theta)]^2 + 2K_1(n, \theta)\, K_2(n, \theta) \\ K_1(n^2, 2\theta) &= [K_2(n, \theta)]^2 + 2K_0(n, \theta)\, K_1(n, \theta) \\ K_2(n^2, 2\theta) &= [K_1(n, \theta)]^2 + 2K_2(n, \theta)\, K_0(n, \theta) \end{aligned}\right\}. \tag{13}$$

Some special values of the K functions. By (7)

$$\left.\begin{aligned} K_0(n, -\theta) &= K_0(n, \theta) \\ K_1(n, -\theta) &= K_2(n, \theta) \\ K_2(n, -\theta) &= K_1(n, \theta) \end{aligned}\right\}. \tag{14}$$

$$\left.\begin{aligned} K_0(n,0) &= \tfrac{1}{3}(1+2n), \quad K_0(1,0) = 1, \\ K_1(n,0) &= \tfrac{1}{3}(1-n), \quad K_1(1,0) = 0, \\ K_2(n,0) &= \tfrac{1}{3}(1-n), \quad K_2(1,0) = 0, \end{aligned}\right\} \tag{15}$$

$$\left.\begin{aligned} K_0\left(-1,\frac{\pi}{3}\right) &= 0, \quad K_0\left(1,\frac{2\pi}{3}\right) = 0, \\ K_1\left(-1,\frac{\pi}{3}\right) &= 0, \quad K_1\left(1,\frac{2\pi}{3}\right) = 1, \\ K_2\left(-1,\frac{\pi}{3}\right) &= 1, \quad K_2\left(1,\frac{2\pi}{3}\right) = 0. \end{aligned}\right\} \tag{16}$$

$$\left.\begin{aligned} K_0\left(n,\frac{\pi}{2}\right) = K_1\left(n,\frac{\pi}{6}\right) &= K_2\left(n,\frac{5\pi}{6}\right) = \tfrac{1}{3} \\ K_1\left(n,\frac{\pi}{2}\right) = K_0\left(n,\frac{\pi}{6}\right) &= K_1\left(n,\frac{5\pi}{6}\right) = \tfrac{1}{3}(1+n\sqrt{3}) \\ K_2\left(n,\frac{\pi}{2}\right) = K_2\left(n,\frac{\pi}{6}\right) &= K_0\left(n,\frac{5\pi}{6}\right) = \tfrac{1}{3}(1-n\sqrt{3}) \end{aligned}\right\}. \tag{17}$$

By (10) and (16)

$$\omega^k = \left[F\left(1,\frac{2\pi}{3}\right)\right]^k = F\left(1,\frac{2k\pi}{3}\right), \tag{18}$$

so that ω^k is an operator that moves e_0 in the circumscribing circle through k times the arc of 120°. Hence the equation of this circle may be written

$$p = \omega^x e_0. \tag{19}$$

We may also write the equation of a logarithmic spiral through e_0, as

$$p = F(n^x, x\theta). \tag{20}$$

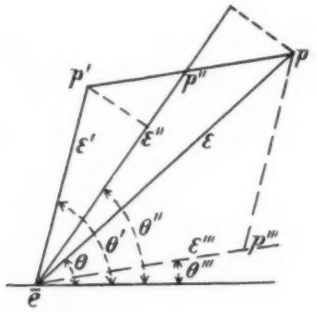
We have also the following relations

$$K_0(n,\theta) = K_1\left(n,\theta+\frac{2\pi}{3}\right) = K_2\left(n,\theta-\frac{2\pi}{3}\right)$$

$$K_1(n,\theta) = K_2\left(n,\theta+\frac{2\pi}{3}\right) = K_0\left(n,\theta-\frac{2\pi}{3}\right)$$

$$K_2(n,\theta) = K_0\left(n,\theta+\frac{2\pi}{3}\right) = K_1\left(n,\theta-\frac{2\pi}{3}\right)$$

Sums and Differences of the K Functions.



Let

$$p = \sum_{0}^{2} xe\,, \quad p' = \sum_{0}^{2} x'e\,;$$

$$\therefore \qquad p'' = \Sigma \left[\frac{x + x'}{2} \,.\, e \right]$$

is the bisecting point of pp'.

Let $Tēp = na$, $Tēp' = n'a$, $Tēp'' = n''a$, and $p - ē = \varepsilon$, $p' - ē = \varepsilon'$, $p'' - e = \varepsilon''$; then

$$T^2ep'' = n''^2a^2 = \tfrac{1}{4}(\varepsilon + \varepsilon')^2 = \tfrac{1}{4}(\varepsilon^2 + \varepsilon'^2 + 2\varepsilon\,\varepsilon')$$

$$= \frac{a^2}{4}(n^2 + n'^2 + 2nn'\cos(\theta - \theta'))$$

$$\therefore \qquad n'' = \tfrac{1}{2}\sqrt{n^2 + n'^2 + 2nn'\cos(\theta - \theta')}\,. \tag{21}$$

Again

$$an\sin(\theta'' - \theta) = an'\sin(\theta' - \theta'')\,,$$

or

$$n(\sin\theta''\cos\theta - \cos\theta''\sin\theta) = n'(\sin\theta'\cos\theta'' - \cos\theta'\sin\theta'')\,,$$

whence

$$\tan\theta'' = \frac{n\sin\theta + n'\sin\theta'}{n\cos\theta + n'\cos\theta'}\,. \tag{22}$$

If $n' = n$, (21) and (22) become

$$n'' = n\cos\frac{\theta - \theta'}{2} \quad \text{and} \quad \theta'' = \frac{\theta + \theta'}{2}\,, \tag{23}$$

while, if $\theta' = \theta$,

$$n'' = \tfrac{1}{2}(n + n'), \quad \text{and} \quad \theta'' = \theta\,. \tag{24}$$

We have therefore in general

$$K(n,\theta) + K(n',\theta') = 2K\left[\tfrac{1}{2}\sqrt{n^2 + n'^2 + 2nn'\cos(\theta-\theta')},\ \tan^{-1}\frac{n\sin\theta + n'\sin\theta'}{n\cos\theta + n'\cos\theta'}\right]. \quad (25)$$

When $n' = n$

$$K(n,\theta) + K(n,\theta') = 2K\left[n\cos\frac{\theta-\theta'}{2},\ \frac{\theta+\theta'}{2}\right], \quad (26)$$

and when $\theta' = \theta$

$$K(n,\theta) + K(n',\theta) = 2K\left[\frac{n+n'}{2},\ \theta\right]. \quad (27)$$

Subtracting we have

$$p - p' = \varepsilon''' = p''' - e = \Sigma(x - x')e;$$

so that

$$p''' = \Sigma K(n''',\theta''')e = \Sigma(x-x')e + \tfrac{1}{3}\Sigma e = \Sigma(x - x' + \tfrac{1}{3})e$$
$$= \Sigma[(K(n,\theta) - K(n',\theta') + \tfrac{1}{3})e];$$

$$\therefore \quad K(n,\theta) - K(n',\theta') = K(n''',\theta''') - \tfrac{1}{3}. \quad (28)$$

But

$$T\varepsilon''' = T(\varepsilon - \varepsilon') = \sqrt{\varepsilon^2 + \varepsilon'^2 - 2\varepsilon\varepsilon'},$$

$$\therefore \quad n''' = \sqrt{n^2 + n'^2 - 2nn'\cos(\theta-\theta')}$$

and

$$\theta''' = \tan^{-1}\frac{n\sin\theta - n'\sin\theta'}{n\cos\theta - n'\cos\theta'},$$

hence

$$K(n,\theta) - K(n',\theta')$$
$$= K\left[\sqrt{n^2 + n'^2 - 2nn'\cos(\theta-\theta')},\ \tan^{-1}\frac{n\sin\theta - n'\sin\theta'}{n\cos\theta - n'\cos\theta'}\right] - \tfrac{1}{3}. \quad (29)$$

When $n' = n$

$$K(n,\theta) - K(n,\theta') = K\left[2n\sin\frac{\theta-\theta'}{2},\ \frac{\theta+\theta'+\pi}{2}\right] - \tfrac{1}{3}. \quad (30)$$

When $\theta' = \theta$

$$K(n,\theta) - K(n',\theta) = K[n - n',\theta] - \tfrac{1}{3}. \quad (31)$$

We have thus a trigonometry of the K functions analogous to that of the circular functions.

Besides whatever interest these relations may have due to their analogies, they might prove of use when working with a point system in plane space, for comparing the location of points by weights with that by angle and distance.

CRITERIA FOR NODES IN DUPIN'S CYCLIDES, WITH A CORRESPONDING CLASSIFICATION.*

By Dr. Virgil Snyder, Ithaca, N. Y.

Since the introduction of generalized coördinates, new surfaces have engaged the attention of mathematicians, and known ones have been found to possess many new and interesting properties. Prominent among these are the cyclides, surfaces of the fourth order defined as the envelope of a system of spheres, fulfilling either two or three conditions, and especially Dupin's cyclide, defined as the envelope of a singly infinite system of spheres which touch three given ones.

In the classification of cyclides by Loria† the cyclide of Dupin only appears as a single type: the specific forms of these surfaces are not considered. Loria uses five spheres to define his system of (point) coördinates, discussed as "coördinées pentaspheriques" by Darboux‡ and the general cyclide is expressed by a quadratic equation.

By using the hexaspherical coördinates introduced by Lie§ the sphere, not the point, is regarded as the generating element. There is a close analogy between this geometry and the geometry of the straight line as introduced by Plücker,‖ Dupin's cyclide playing exactly the same rôle in the former, as the hyperboloid of one nappe in the latter. In a recent paper¶ the writer gave the details of this analogy for the linear equation. An imaginary transformation is there employed which makes it impossible to determine criteria for reality of any particular properties.

In this paper it is proposed to derive the criteria for nodes in Dupin's Cyclides, using the symbols and nomenclature employed in the previous paper, but without making any further reference to line geometry.

It will be sufficient to give an outline of the notation and terms used.

Let

$$(\mu - \nu)(x^2 + y^2 + z^2) - 2\xi xt - 2\eta yt - 2\zeta zt + (\mu + \nu)t^2 = 0 \qquad (1)$$

* Read at the Buffalo meeting of the American Mathematical Society.

† G. Loria: Recerche intorno alla geometria della spera e loro applicazione, . . . Memoire della Accademia Reale di Torino, II, 36, 1884.

‡ G. Darboux: Sur une Classe remarquable de Courbes et de Surfaces . . . Paris, 1873. 2d Edition, 1896.

§ S. Lie: Ueber Complexe, insbesondere über Linieri une Kugelcomplexe und ihre Anwendung auf die Theorie der Differentialgleichungen." Math. Ann. BV, 1872.

‖ J. Plucker: Neue Geometrie des Raumes, mit der geraden Linie as Raum element. Leipzig, 1868–9.

¶ Ueber die linearen Complexe der Lie'schen Kugelgeometrie. Göttingen, 1895.

be the homogeneous equation of a sphere; denote the radius by $\frac{\lambda}{\mu - \nu}$, then the quadratic relation

$$\Pi = \xi^2 + \eta^2 + \zeta^2 - \lambda^2 - \mu^2 + \nu^2 = 0 \tag{2}$$

exists between these quantities, which may be called the *six homogeneous coördinates of the sphere;* conversely, any six quantities which fulfil this relation, are sufficient to fix a sphere.

Two spheres ξ, η, ζ; $\xi', \eta', \zeta' \ldots$ cut each other at an angle φ defined by the equation

$$\xi\xi' + \eta\eta' + \zeta\zeta' - \lambda\lambda' \cos\varphi - \mu\mu' + \nu\nu' = 0\,. \tag{3}$$

Now consider a general linear equation with real coefficients, between $\xi, \eta, \ldots,$

$$\Psi = a\xi + b\eta + c\zeta + d\lambda + e\mu + f\nu = 0\,. \tag{4}$$

The sphere whose coördinates satisfy this equation constitute a *linear spherical complex.* Make (4) identical with (3); by putting

$$a = \xi'\,;\ b = \eta'\,;\ c = \zeta'\,,\ d = -\lambda' \cos\varphi\,;\ e = -\mu'\,,\ f = \nu'\,.$$

This can always be done for the only condition required of $\xi', \eta', \ldots$ is $\Pi = 0$ and $\cos\varphi$ is assignable; hence

A linear spherical complex consists of the ∞^3 spheres which cut a fixed sphere at a constant angle.

The fixed sphere is called the *fundamental sphere,* and the constant angle, the *angle of the complex.*

The radius of the fundamental sphere, ρ, is defined by the equation

$$\rho = \frac{\pm\sqrt{a^2 + b^2 + c^2 - e^2 + f^2}}{e + f}, \tag{5}$$

and the angle of the complex, φ, by the equation

$$\cos\varphi = \frac{d}{\sqrt{a^2 + b^2 + c^2 - e^2 + f^2}}. \tag{6}$$

The expression

$$S = a^2 + b^2 + c^2 - d^2 - e^2 + f^2 \tag{7}$$

is called the *invariant* of the complex. When $\varphi = 0$, in (6) $S = 0$, and the complex becomes *special;* it consists of all the spheres which touch the sphere (a, b, c, d, e, f).

A point may be regarded as a sphere with a vanishing radius (point sphere), and a plane as a sphere with an infinite radius. There are ∞^3 spheres

having a vanishing radius, which is expressed by the condition $\lambda = 0$; there are ∞^3 spheres having an infinite radius, which is expressed by the condition $\mu - \nu = 0$ [cf. eq. (1)], hence

The points and planes of space form each a linear spherical complex; their equations are, respectively,

$$\lambda = 0 \text{ and } \mu - \nu = 0 . \tag{8}$$

The spheres common to two complexes, $\psi_1 = 0$, $\psi_2 = 0$ form a *linear spherical congruence.* The congruence evidently belongs to every complex of the pencil

$$\psi_1 + \varkappa\psi_2 = 0 \tag{9}$$

where $\varkappa$ is a varying parameter. Among these complexes are two special ones, defined by the equation

$$S_{11} + 2\varkappa S_{12} + \varkappa^2 S_{22} = 0 , \tag{10}$$

where S_{11}, S_{22} are the invariants of ψ_1, ψ_2, and

$$S_{12} = a_1a_2 + b_1b_2 + c_1c_2 - d_1d_2 - e_1e_2 + f_1f_2$$

is the *simultaneous invariant* of ψ_1, ψ_2.

The fundamental spheres of the two special complexes contained in the pencil (9) are real, coincident or imaginary, according as

$$\begin{vmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{vmatrix} \quad [S_{ik} = S_{ki}] \tag{11}$$

is negative, zero, or positive.

The spheres belonging to three complexes ψ_1, ψ_2, ψ_3 form a linear spherical *series;* they evidently belong to every complex of the sheaf

$$\psi_1 + \varkappa\psi_2 + \tau\psi_3 = 0 , \tag{12}$$

$\varkappa$, τ being varying parameters. In this sheaf are ∞^1 special complexes, determined by those values of $\varkappa$, τ which satisfy the equation

$$S_{11} + 2\varkappa S_{12} + \varkappa^2 S_{22} + 2\varkappa\tau S_{23} + 2\tau S_{13} + \tau^2 S_{33} = 0 . \tag{13}$$

The fundamental spheres of these special complexes are the *directrices* of the series. Every sphere of the series touches all the directrices, and conversely, but no two spheres of the same system can touch each other. This

configuration is a *Dupin's cyclide;* the centres of the spheres belonging to the series lie on one focal line, and those of the directrices on the other. These focal lines are conics in planes perpendicular to each other (planes of symmetry of the surface) the vertices of each being the foci of the other.*

The discriminant of the invariant (13)

$$\Delta \quad \begin{matrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{matrix}, \tag{14}$$

being of odd order, has no importance attached to its sign. When $\Delta = 0$, the cyclide reduces to a gauche quadrilateral of absolute lines.†

There is a finite number of spheres belonging to four complexes, $\psi_1 = 0$, $\psi_2 = 0$, $\psi_3 = 0$, $\psi_4 = 0$; there being four linear and one quadratic equation (2), defining ξ, η, . . . the number is seen to be two.‡

These spheres may be real or imaginary; they belong to the ∞^3 complexes of the group

$$\psi_1 + \varkappa\psi_2 + \tau\psi_3 + \sigma\psi_4 = 0\,. \tag{15}$$

The invariant of this complex is a function of $\varkappa$, τ, σ. The complex becomes special for values of $\varkappa$, τ, σ which lie on a quadric surface, whose discriminant is

$$D = \begin{matrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{21} & S_{22} & S_{23} & S_{24} \\ S_{31} & S_{32} & S_{33} & S_{34} \\ S_{41} & S_{42} & S_{43} & S_{44} \end{matrix}. \tag{16}$$

Analogous to Eq. (11), the two spheres common to four complexes are real, coincident or imaginary, according as D is negative, zero, or positive.

The truth of this statement is not so obvious as in case of Eq. 9, but an elementary proof, requiring nothing but algebra, is as follows:

* See Salmon: Geometry of Three Dimensions, 4th Edition, p. 535.

† Compare "Ueber der linearen Complexe," . . . p. 37.

‡ It must be borne in mind that spheres cutting the fundamental sphere *from within* at the angle φ, do not belong to the same complex as those which cut it at the same angle *from without*. The same rule applies to internal and external contact.

The coördinates of the two spheres, common to four complexes, being determined by the values of ξ, η ... which satisfy

$$\psi = a\xi + b\eta + c\zeta + d\lambda + e\mu + f\nu = 0$$
$$\psi_1 = a_1\xi + b_1\eta + c_1\zeta + d_1\lambda + e_1\mu + f_1\nu = 0$$
$$\psi_2 = a_2\xi + b_2\eta + c_2\zeta + d_2\lambda + e_2\mu + f_2\nu = 0$$
$$\psi_3 = a_3\xi + b_3\eta + c_3\zeta + d_3\lambda + e_3\mu + f_3\nu = 0$$

and the identical relation

$$\Pi = \xi^2 + \eta^2 + \zeta^2 - \lambda^2 - \mu^2 + \nu^2 = 0\,,$$

are dependent upon the roots of a quadratic equation. These coördinates can only be real when the discriminant of this quadratic equation is negative.

By putting one coördinate (ν) equal to 1 and solving the first four equations for four coördinates (ξ, η, ζ, λ) in terms of the other one (μ), then substituting their values in Π, the desired quadratic will be obtained. It will be necessary to show that the discriminant of this equation equals D multiplied by a square factor, the form of which depends upon the coördinate retained in Π.

Let

$$\Delta \equiv \begin{vmatrix} a & b & c & d \\ a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{vmatrix}$$

then

$$\xi = \frac{\begin{vmatrix} e\mu + f & b & c & d \\ e_1\mu + f_1 & b_1 & c_1 & d_1 \\ e_2\mu + f_2 & b_2 & c_2 & d_2 \\ e_3\mu + f_3 & b_3 & c_3 & d_3 \end{vmatrix}}{\Delta}.$$

This can be expressed by the sum of two determinants, and to avoid complicated terms, it is well to introduce some abbreviations.

Let $\begin{bmatrix} \Delta \\ a \end{bmatrix}_e$ be what Δ becomes when a, a_1, a_2, a_3 have been replaced by e, e_1, e_2, e_3, with similar notation for the other terms, then

$$\Delta^2\xi^2 = \mu^2\begin{bmatrix} \Delta \\ a \end{bmatrix}_e^2 + 2\mu\begin{bmatrix} \Delta \\ a \end{bmatrix}_e\begin{bmatrix} \Delta \\ a \end{bmatrix}_f + \begin{bmatrix} \Delta \\ a \end{bmatrix}_f^2$$

and similar terms for η, ζ, λ. Substituting them in Π, it becomes

$$\left\{\left[\begin{matrix}\Delta\\ a\end{matrix}\right]_e^2+\left[\begin{matrix}\Delta\\ b\end{matrix}\right]_e^2+\left[\begin{matrix}\Delta\\ c\end{matrix}\right]_e^2-\left[\begin{matrix}\Delta\\ d\end{matrix}\right]_e^2-\Delta^2\right\}\mu^2$$
$$+2\left\{\left[\begin{matrix}\Delta\\ a\end{matrix}\right]_e\left[\begin{matrix}\Delta\\ a\end{matrix}\right]_f+\left[\begin{matrix}\Delta\\ b\end{matrix}\right]_e\left[\begin{matrix}\Delta\\ b\end{matrix}\right]_f+\left[\begin{matrix}\Delta\\ c\end{matrix}\right]_e\left[\begin{matrix}\Delta\\ c\end{matrix}\right]_f-\left[\begin{matrix}\Delta\\ d\end{matrix}\right]_e\left[\begin{matrix}\Delta\\ d\end{matrix}\right]_f\right\}\mu$$
$$+\left\{\left[\begin{matrix}\Delta\\ a\end{matrix}\right]_f^2+\left[\begin{matrix}\Delta\\ b\end{matrix}\right]_f^2+\left[\begin{matrix}\Delta\\ c\end{matrix}\right]_f^2-\left[\begin{matrix}\Delta\\ d\end{matrix}\right]_f^2+\Delta^2\right\}=0\,.$$

The discriminant of this equation can, after a few reductions, be brought to the form

$$\left[\left[\begin{matrix}\Delta\\ a\end{matrix}\right]_e\left[\begin{matrix}\Delta\\ b\end{matrix}\right]_f-\left[\begin{matrix}\Delta\\ a\end{matrix}\right]_f\left[\begin{matrix}\Delta\\ b\end{matrix}\right]_e\right]^2+\left[\left[\begin{matrix}\Delta\\ a\end{matrix}\right]_e\left[\begin{matrix}\Delta\\ c\end{matrix}\right]_f-\left[\begin{matrix}\Delta\\ a\end{matrix}\right]_f\left[\begin{matrix}\Delta\\ c\end{matrix}\right]_e\right]^2$$
$$+\left[\left[\begin{matrix}\Delta\\ b\end{matrix}\right]_e\left[\begin{matrix}\Delta\\ c\end{matrix}\right]_f-\left[\begin{matrix}\Delta\\ b\end{matrix}\right]_f\left[\begin{matrix}\Delta\\ c\end{matrix}\right]_e\right]^2-\left[\left[\begin{matrix}\Delta\\ a\end{matrix}\right]_e\left[\begin{matrix}\Delta\\ d\end{matrix}\right]_f-\left[\begin{matrix}\Delta\\ a\end{matrix}\right]_f\left[\begin{matrix}\Delta\\ d\end{matrix}\right]_e\right]^2$$
$$-\left[\left[\begin{matrix}\Delta\\ b\end{matrix}\right]_e\left[\begin{matrix}\Delta\\ d\end{matrix}\right]_f-\left[\begin{matrix}\Delta\\ b\end{matrix}\right]_f\left[\begin{matrix}\Delta\\ d\end{matrix}\right]_e\right]^2-\left[\left[\begin{matrix}\Delta\\ c\end{matrix}\right]_e\left[\begin{matrix}\Delta\\ d\end{matrix}\right]_f-\left[\begin{matrix}\Delta\\ c\end{matrix}\right]_f\left[\begin{matrix}\Delta\\ d\end{matrix}\right]_e\right]^2$$
$$+\Delta^2\left[\left[\begin{matrix}\Delta\\ a\end{matrix}\right]_e^2-\left[\begin{matrix}\Delta\\ a\end{matrix}\right]_f^2+\left[\begin{matrix}\Delta\\ b\end{matrix}\right]_e^2-\left[\begin{matrix}\Delta\\ b\end{matrix}\right]_f^2+\left[\begin{matrix}\Delta\\ c\end{matrix}\right]_e^2-\left[\begin{matrix}\Delta\\ c\end{matrix}\right]_f^2-\left[\begin{matrix}\Delta\\ d\end{matrix}\right]_e^2+\left[\begin{matrix}\Delta\\ d\end{matrix}\right]_f^2-\Delta^2\right].$$

On the other hand, D can be expressed as the product of the two rectangular arrays

$$\left\{\begin{matrix}a & b & c & -d & -e & f\\ a_1 & b_1 & c_1 & -d_1 & -e_1 & f_1\\ a_2 & b_2 & c_2 & -d_2 & -e_2 & f_2\\ a_3 & b_3 & c_3 & -d_3 & -e_3 & f_3\end{matrix}\right\}\cdot\left\{\begin{matrix}a & b & c & d & e & f\\ a_1 & b_1 & c_1 & d_1 & e_1 & f_1\\ a_2 & b_2 & c_2 & d_2 & e_2 & f_2\\ a_3 & b_3 & c_3 & d_3 & e_3 & f_3\end{matrix}\right\}$$

which can be developed as the sum of a series of positive and negative squares,

$$D=\left[-\Delta^2-\left[\begin{matrix}\Delta\\ d\end{matrix}\right]_e^2+\left[\begin{matrix}\Delta\\ d\end{matrix}\right]_f^2+\left[\begin{matrix}\Delta\\ c\end{matrix}\right]_e^2-\left[\begin{matrix}\Delta\\ c\end{matrix}\right]_f^2+\left[\begin{matrix}\Delta\\ b\end{matrix}\right]_e^2-\left[\begin{matrix}\Delta\\ b\end{matrix}\right]_f^2+\left[\begin{matrix}\Delta\\ a\end{matrix}\right]_e^2-\left[\begin{matrix}\Delta\\ a\end{matrix}\right]_f^2\right]$$
$$-(a_1b_2e_2f_3)^2-(ac_1e_2f_3)^2-(bc_1e_2f_3)^2+(ad_1e_2f_3)^2+(bd_1e_2f_3)^2+(cd_1e_2f_3)^2\,.$$

The first nine terms are exactly the same as the coefficient of Δ^2 in the discriminant, and on account of the identity

$$(eb_1c_2d_3)(ab_1c_2f_3) - (fb_1c_2d_3)(ab_1c_2e_3) - (ab_1c_2d_3)(ab_1e_2f_3) = 0^*$$

and similar expressions for the other five terms, the entire discriminant is $\Delta^2 D$. The negative of this expression is the quantity under the radical sign, in the roots of the quadratic defining μ, and all the other coördinates being linearly expressible in terms of μ, hence; *the spheres common to four linear spherical complexes are real, coincident, or imaginary, according as the combinant of the four complexes is negative, zero, or positive.*†

These properties will now be made use of to classify Dupin's cyclides, according to the arrangement of their nodes and those tangent-planes which are touched by every sphere of the other generation.

At least one of the focal lines must extend to infinity; these points at infinity are the centers of spheres which must touch every sphere belonging to the other generation of the surface, that is, they are planes. With the exception of the case where both focal lines become parabolas, one is an ellipse, the other a hyperbola. Moreover, the direction of the asymptotes indicate the position of the generating planes; in the hyperbola there are two such directions; in the parabola they coincide and in the ellipse become imaginary.

Either generation of the surface may contain points, but not both can contain them; for as every sphere of each generation must touch all the spheres of the other one, if both contained points, the focal lines would have points in common, which is impossible.

The presence of these point-spheres in either generation indicates that each such point is a node of the surface. To apply the criteria obtained above, consider the three complexes defining a cyclide, and the complex of point-spheres, $\lambda = 0$.

The series (12) contains two points, which are real, coincident, or imaginary, according as

$$A = \begin{vmatrix} S_{11} & S_{12} & S_{13} & -d_1 \\ S_{21} & S_{22} & S_{23} & -d_2 \\ S_{31} & S_{32} & S_{33} & -d_3 \\ -d_1 & -d_2 & -d_3 & -1 \end{vmatrix} \qquad (17)$$

* See Muir's Determinants, p. 139.

† Evidently this same form (with e, a, f^2 negative, to make all coördinates real, and ν^2 negative in Π) holds for two lines common to four linear line complexes. That the lines coincide when D vanishes was known by *Plücker* (and stated by *Klein* in his memoir, "Differentialgleichungen der Liniengeometrie," Math. Ann. Vol. 5.)

is negative, zero, or positive. In the same manner, the series contains two planes, which are real, coincident, or imaginary, according as

$$N = \begin{vmatrix} S_{11} & S_{12} & S_{13} & -(e_1 + f_1) \\ S_{21} & S_{22} & S_{23} & -(e_2 + f_2) \\ S_{31} & S_{32} & S_{33} & -(e_3 + f_3) \\ -(e_1 + f_1) & -(e_2 + f_2) & -(e_3 + f_3) & 0 \end{vmatrix} \tag{18}$$

is negative, zero, or positive.

Two points are contained among the directrices; they are defined by those values of $\varkappa$, τ which satisfy (13) and

$$d_1 + \varkappa d_2 + \tau d_3 = 0 . \tag{19}$$

for the radius $\rho_\varkappa$, of the fundamental circle belonging to the complex $\psi_\varkappa$ is given by the equation

$$\rho_\varkappa = \frac{\sqrt{(a_1+\varkappa a_2+\tau a_3)^2+(b_1+\varkappa b_2+\tau b_3)^2+(c_1+\varkappa c_2+\tau c_3)^2-(e_1+\varkappa e_2+\tau e_3)^2+(f_1+\varkappa f_2+\tau f_3)}}{(e_1 + \varkappa e_2 + \tau e_3) + (f_1 + \varkappa f_2 + \tau f_3)} \tag{20}$$

where $\psi_\varkappa = \psi_1 + \varkappa\psi_2 + \tau\psi_3$. (See Eq. 5.) This circle being a directrix, the complex is a special one, so the numerator can be replaced by $d_1 + \varkappa d_2 + \tau d_3$: finally, when this directrix reduces to a point, the radius vanishes, which is expressed in (19).

The values of $\varkappa$, τ common to (13) and (19) are real, coincident, or imaginary, according as

$$M = \begin{vmatrix} S_{11} & S_{12} & S_{13} & d_1 \\ S_{21} & S_{22} & S_{23} & d_2 \\ S_{31} & S_{32} & S_{33} & d_3 \\ d_1 & d_2 & d_3 & 0 \end{vmatrix} \tag{21}$$

is positive, zero, or negative.*

Since either generation may contain points, but not both, it follows that *every Dupin's cyclide has four nodes, of which at least two are imaginary*, and all may be.

Two planes are contained among the directrices; they are found by substituting those values of $\varkappa$, τ in (12) that satisfy (13) and

$$e_1 + f_1 + \varkappa (e_2 + f_2) + \tau (e_3 + f_3) = 0 , \tag{22}$$

* This equation expresses whether the line (19) intersects the conic (13) in real, coincident, or imaginary points in the $\varkappa$, τ plane. See Salmon's Conic Sections, p. 267.

this last equation being analogous to (19), derived from (20) by making the radius of a directrix infinite.

The values of $\varkappa$, τ are real, coincident, or imaginary, according as

$$N \equiv \begin{vmatrix} S_{11} & S_{12} & S_{13} & e_1 + f_1 \\ S_{21} & S_{22} & S_{23} & e_2 + f_2 \\ S_{31} & S_{32} & S_{33} & e_3 + f_3 \\ e_1 + f_1 & e_2 + f_2 & e_3 + f_3 & 0 \end{vmatrix} \tag{23}$$

is positive, zero, or negative.

This determinant is identical with (18); when one generation has real planes, those in the other become imaginary except when both pairs become coincident. This is only a corroboration of what was seen before, by geometrical considerations.

It will now be necessary to define the cyclides:

A cyclide having parabolas for focal lines is called a *parabolic cyclide;* this surface extends to infinity and is of the third degree. It may have no nodes, two nodes, or a double node. In each case two straight lines can be drawn on the surface, at right angles to each other.

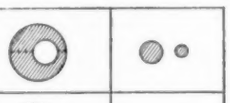
All other cyclides are closed surfaces of the fourth degree. There are three general types:

A cyclide having no nodes is called a *ring-cyclide;* the anchor ring is an illustration.

The nodal cyclides are of two forms, according as the points and planes belong to the same, or different generations; the former are called *spindle cyclides;* they consist of two sheets wound about each other, or of one spindle inside another spindle. The latter are called *horn-cyclides;* they consist of two crescents, with their horns together.

From the physical appearances of these surfaces, there is no difference between series and directrices, so that each must be considered in two ways. For example, the horn-cyclide may have nodes in the first generation (series) and planes in the second (directrices) or conversely. The analytical criteria for the two forms are not the same, and will here be considered independently.

Now, since for $\rho = 0$ and $\rho = \infty$, no attention need be paid to the sign of ρ^2 (spheres with real equations, but imaginary radii), the form of a surface defined by three complexes, is determined by the signs of Λ, M, N. To assist the mind in forming a picture, the traces of the surfaces on the two planes of symmetry will be added in the following table.

A	M	N	Surface.	Traces on planes of symmetry.
+	−	−	Ring-cyclide.	
+	−	+		
−	−	+	Horn-cyclide.	
+	+	−		
+	+	+	Spindle-cyclide.	
−	−	−		
0	+	+	Cuspidal-cyclide.	
−	0	−		
0	+	−	Pinch-cyclide.	
−	0	+		
+	−	0	Parabolic Ring-cyclide.	
+	−	0		
+	+	0	Parabolic Binoidal-cyclide.	
−	−	0		
0	+	0	Parabolic Cuspidal-cyclide.	
−	0	0		

This completes the solution of the general problem.

Cyclides of revolution have a circle for one focal line, and a straight line from the other. They must be either ring-cyclides or spindle-cyclides, as the spheres whose centers lie on the circle must all be of the same radius.

If this be the series, the sheaf (12) can be reduced to the form

$$\varphi_1 + \varkappa\varphi_2 + \tau\varphi_3 \equiv m\lambda + n(\mu - \nu)$$

for some values of $\varkappa, \tau$. The determinants in the coefficients, which vanish to satisfy this equation, give the necessary and sufficient condition that the series is one of rotation.

If the second generation has a circle for a focal-line, all the directrices must have the same radius ρ, so

$$\rho = \frac{d_1 + \varkappa d_2 + \tau d_3}{e_1 + f_1 + \varkappa(e_2 + f_2) + \tau(e_3 + f_3)}$$

must be independent of the value of x, τ, or

$$d_1 : d_2 : d_3 = (e_1 + f_1) : (e_2 + f_2) : (e_3 + f_3) .$$

The *cone* is a cyclide having one generation composed of planes; ($m = 0$ in the above form) or

$$e_1 + f_1 = e_2 + f_2 = e_3 + f_3 = 0 .$$

The *circle* is a cyclide having one generation composed of points.

$$n = 0, \text{ or } d_1 = d_2 = d_3 = 0 .$$

The *right line* is a cyclide, with one generation of points and one of planes. Combination of above criteria. These forms were treated in my paper named above.

CORNELL UNIVERSITY, *Aug.*, 1896.

ON THE CONGRUENCES OF RAYS (3, 1) AND (1, 3).

By Dr. Arnold Emch, Biel, Switzerland.

1. In No. 1, Vol. 10, of the *Annals of Mathematics*, I have referred to a certain theorem concerning the projective transformations of the plane which has served Prof. Newson and myself as a means to treat the projective groups of the plane.* The theorem has been stated as follows:

To conics K and K' in a plane tangent to the same straight line l determine a projective transformation.

To find the corresponding point to a point P, draw from P the two tangents to the conic K, and from the points where they intersect the line L two tangents to the conic K'. Where these tangents intersect each other is the required point P'.

In connection with the theory based upon it in the papers referred to above this theorem appears rather as an artifice and it is the purpose of this note to discuss the congruences of rays (3, 1) and (1, 3) which naturally lead to the foregoing theorem and its dualistic interpretation.

Said congruences are contained in the *Tetrahedral Complex* which, as it is well known, is formed by all rays whose four points of intersection with the four planes of a tetrahedron determine a constant anharmonic ratio. To study the relation between these congruences and the tetrahedral complex, or all the projective transformations in the plane and the complex, it seems advisable to depart from Lie's differential equation of the tetrahedral complex:

$$(b-c)\,xdydz + (c-a)\,ydzdx + (a-b)\,zdxdy = 0\,,$$

where

$$\frac{-1}{b-c} = \frac{k}{c-a} = \frac{1-k}{a-b}\,,$$

k being a constant.†

In our special problem, however, we may conveniently avail ourselves of

* Arnold Emch: Projective Groups of Perspective collineations in the plane (Dissertation); Kans. Univ. Quart., Vol. IV, Nos. 2 and 4; Vol. V, No. 1. H. B. Newson: Projective Groups. Kans. Univ. Quart., Vol. IV, No. 2.

† Sophus Lie: Geometrie der Berührungs transformationen, Vol. I, page 326. Teubner, Leipzig.

some facts about congruences of rays as they have been stated in numerous works.*

2. A congruence of rays of the 3. order and 1. class, which for convenience is called a congruence (3, 1), is perfectly determined by the system of right lines connecting all the corresponding points of two collinear planes Σ and Σ' in space. The focal surface of this congruence is a developable surface of the third class and the fourth order and its edge of regression is a curve of the 3. order and 4. class. Designating by A, B, C in Σ and A', B', C' in Σ' three linear expressions in plane-coordinates which represent three corresponding points of the collineation between Σ and Σ', and by α, β, γ three arbitrary parameters, the developable surface may be considered as the envelope of the host (S-charr) of surfaces of the second class (ruled hyperboloids)

$$\alpha(BC' - B'C) + \beta(CA' - C'A) + \gamma(AB' - A'B) = 0\,, \qquad (1)$$

or

$$\begin{vmatrix} A & A' & \alpha \\ B & B' & \beta \\ C & C' & \gamma \end{vmatrix} = 0\,. \qquad (2)$$

Each of the equations $BC' - B'C = 0$, $CA' - C'A = 0$, $AB' - A'B = 0$, represents a surface of the second class inscribed to the developable surface of the 3. class. Taking another surface of the host (1), different from (2) when α, β, γ are fixed quantities, and represented by the determinant

$$\begin{vmatrix} A & A' & \alpha_1 \\ B & B' & \beta_1 \\ C & C' & \gamma_1 \end{vmatrix} = 0\,, \qquad (3)$$

it can be shown that (2) and (3) have, besides being inscribed to the developable surface of the 3. class, a generatrix in common which is determined by the equations

$$\begin{vmatrix} A & \alpha & \alpha_1 \\ B & \beta & \beta_1 \\ C & \gamma & \gamma_1 \end{vmatrix} = 0\,, \qquad \begin{vmatrix} A' & \alpha & \alpha_1 \\ B' & \beta & \beta_1 \\ C' & \gamma & \gamma_1 \end{vmatrix} = 0\,.$$

* W. Fiedler: Geometrie der Lage, § 78, p. 536. Darboux: La Théorie Générale des Surfaces, Vol. II, Chapt. I.

In fact, these determinants may be written

$$A = \rho\alpha + \rho_1\alpha_1\,, \quad B = \rho\beta + \rho_1\beta_1\,, \quad C = \rho\gamma + \rho_1\gamma_1\,;$$
$$A' = \sigma\alpha + \sigma_1\alpha_1\,, \quad B' = \sigma\beta + \sigma_1\beta_1\,, \quad C' = \sigma\gamma + \sigma_1\gamma_1\,,$$

and these expressions satisfy equations (2) and (3). Conversely, it is seen that through every line of the congruence two surfaces of the second class of the host (1) may be passed. But to make the statement general, there is a single infinite number of surfaces of the host (1) passing through any line of the congruence; and they are represented by

$$\begin{vmatrix} A & A' & \alpha \\ B & B' & \beta \\ C & C' & \gamma \end{vmatrix} + \lambda \begin{vmatrix} A & A' & \alpha_1 \\ B & B' & \beta_1 \\ C & C' & \gamma_1 \end{vmatrix} = 0\,. \tag{4}$$

Every line of the congruence has two focal points which are its points of tangency with the focal surface of the congruence. At these points all the surfaces of the pencil (4) are tangent to the developable surface; or the osculating planes at these points are common tangent planes to all the surfaces of the 2. class (6). From the theory of congruences it is also known that through every line of the congruence there are two developable surfaces whose generatrices belong to the congruence. In our case these surfaces become curves of the second class and they are produced by the osculating planes which pass through the line of the congruence. In fact, every osculating plane intersects the developable surface of the curve in space of the 3. order in a tangent, which is to be counted twice, and a conic. The tangents to this conic are the lines of intersection of all the osculating planes with the osculating plane containing the conic, and belong therefore to the congruence.

Through a point in space there are three osculating planes to the curve in space of the 3. order, hence also three lines of the congruence. On the other hand in each plane lies one line of the congruence. The congruence is therefore of the 3. order and 1. class. To sum up we may say:

A congruence of rays (3, 1) *is formed by the system of all lines of intersection between the osculating planes of a curve in space of the* 3. *order.*

Every osculating plane intersects the developable surface in a curve of the second class, or order, and its tangents are formed by the system of intersection-lines of the given osculating plane with all the other osculating planes. Every line of the congruence is a double tangent to the developable surface and is the intersection of two osculating planes. The generatrices of the developable surface belong to the congruence, too.

3. Assume now any two osculating planes Σ and Σ' of the curve of the 3. order and draw their conic-sections K and K' with the developable surface (Fig. 1).

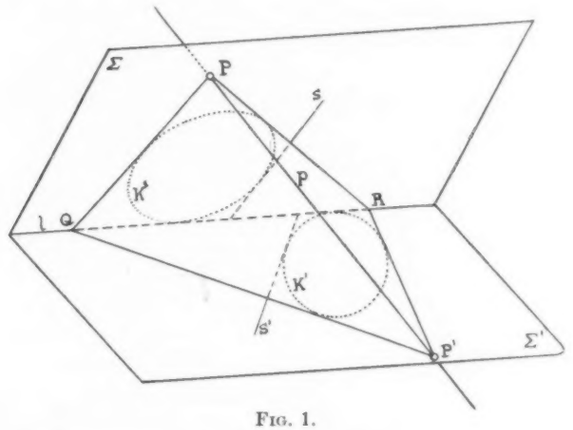


FIG. 1.

The line of intersection l of Σ and Σ', being also a line of the congruence, is tangent to K and K', and so are the generatrices s and s' of the developable surface in Σ and Σ'.

Taking any line p of the congruence, intersecting Σ and Σ' in two points P and P', two osculating planes $PP'Q$ and $PP'R$ pass through p which are also tangent to K and K'. Through p there is an infinite number of surfaces of the second class or order whose generatrices belong to the congruence and intersect Σ and Σ' in two collinear systems of points. Thus, P and P' are two corresponding points in two collinear planes Σ and Σ'.

If we now revolve the plane Σ into the plane Σ' about the line l as an axis we have again reached the proposition as it has been stated at the beginning.

4. For the dualistic interpretation we consider two collinear bundles (Bündel) of planes through the points S and S'. Each two corresponding planes of these bundles intersect each other in a straight line. It is known that all these lines form a congruence of the first order and third class (1, 3), or are bi-secants of a certain curve in space of the 3. order. Choosing any two points on this curve, we can connect them successively with all lines of the congruence and obtain in this manner two collinear bundles of planes. Again, through every line of this congruence two developable surfaces may be passed whose generatrices belong to the congruence. In our case the developable surfaces are two cones of the second order which, evidently, have the ray of the congruence and the curve in space of the 3. order in common. As there are ∞^1 points on this curve and as the line connecting any two of these points

defines a ray of the congruence, and with this two cones of the second order which intersect each other in the curve, we may say that every curve in space of the 3. order may be considered ∞^2 times as the product of two intersecting cones of the second order. The rest of the intersection of these ∞^2 cones forms a congruence of rays (1, 3).

5. The result of the previous paragraph gives us a means to make the dualistic construction for two collinear planes when the straight line is considered as the fundamental element.

Take any ray g of the congruence with the focal points P and P' which lie on the curve in space of the 3. order C_3 and construct the two cones of the second order with the vertices P and P' and intersecting each other in the curve C_3. Intersect these two cones by any plane of a general position and project the whole configuration homographically upon this plane. Without affecting the generality of the result we may assume the plane intersection of the two cones as two circles K and K' corresponding to the cones with the vertices P and P'.

If we let the plane of projection coincide with the plane of the paper, Fig. 2 arises.

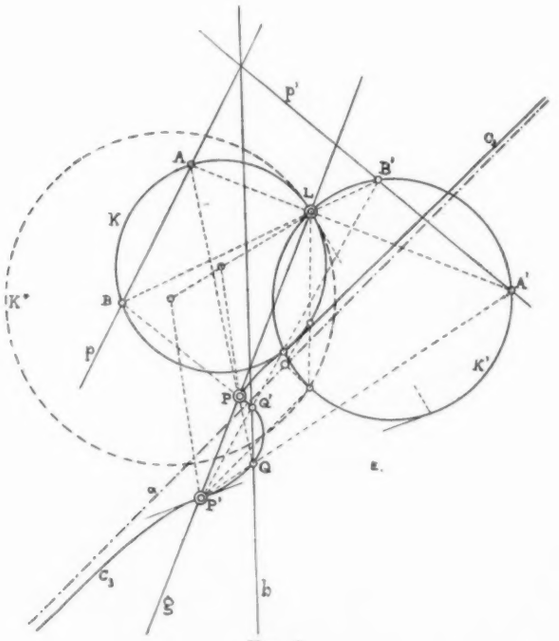


FIG. 2.

L designates the point where the ray g of the congruence pierces the plane of projection and is consequently one of the points of intersection of the

two circles (conics) K and K'. The construction of the curve C_3 is made according to well known principles in descriptive geometry and does not need to be explained here. Take any plane of the bundle through P and suppose it intersects the plane of projection in the line p. The same plane intersects the curve C_3 in two points Q and Q' whose connection-line h is a ray of the congruence. If we therefore draw a plane connecting the ray h with the vertex P', we shall have the corresponding plane in the collinear bundle of planes through P'. It intersects the plane of projection in a line p', the line corresponding to the line p in a collineation. The plane through P and p intersects the cone (P, K) in two generatrices AP and BP on which the points Q and Q' are situated. To find the true location of Q and Q', pass two planes through the ray g of the congruence and the two generatrices AP and BP. These two planes intersect the cone (P', K') in two generatrices $A'P'$ and $B'P'$. Now AP and $A'P'$ lie in the same plane through g and therefore intersect each other in the point Q of the curve. In the same way BP and $B'P'$ intersect each other in the point Q' of the curve. From this we see that the construction of two points of the curve C_3 is identical with the construction of two collinear planes in the collinear bundles through those points. If any line p in the plane of projection is given the corresponding line p' in the same plane is found by connecting A and B (points of intersection of p with K) with L and producing these lines till they intersect the conic K' in two points A' and B'. The line connecting A' with B' is the corresponding line p'.

The dualistic theorem to the one given in § 1. is now proved:

Two conics K and K' in a plane and passing through the same point L determine a projective transformation in this plane.

*To find the line p' corresponding to a line p, connect the points of intersection A and B of p with K, with L and produce these lines to their points of intersection A' and B' with K'. The line connecting A' with B' is the required line p'.**

6. Among the great number of special cases of collineation, perspective is of particular interest. I shall confine myself to the general case of perspective which arises when the curve C_3 representing the congruence (1, 3) consists of a conic C_2 and a straight line g intersecting this conic. The congruence of bi-secants of the curve is in this case of the first order and second class (1, 2). On the ray g we may again choose two points P and P' and pass the two cones through the conic. These cones are tangent to each other and will, therefore, intersect a plane of a general position in two tangent conics K and K', Fig. 3.

* This theorem might be obtained immediately from the first theorem by the principle of reciprocal polars; but it has been preferred to prove it in connection with the congruence (1, 3) as it has been pointed out in the introduction.

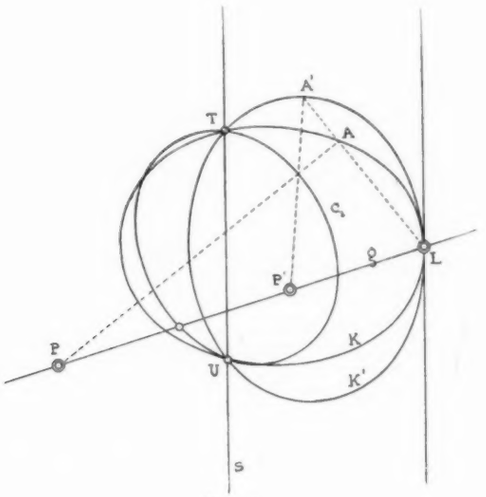


FIG. 3.

As in the general case to a line p intersecting the conic K to which it belongs in two points A and B, the corresponding line p' belonging to the conic K' is obtained by drawing lines connecting A and B with L which when produced intersect the conic K' in two points A' and B'. The line joining A' with B' is the required line p', Fig. 4. From projective geometry is known *that two conics K and K' intersecting each other in the points T and U determine a perspective collineation with the line s joining T with U as an axis and the intersection of two common tangents as a center.*

From this follows that the conics K and K' in Fig. 4 are in perspective with regard to the line s as an axis and the point of tangency L as a center.

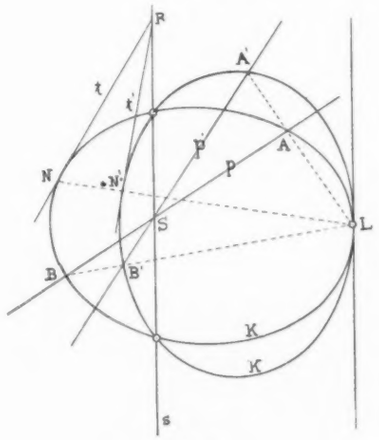


FIG. 4.

Hence, the lines p and p' intersect each other in a point S of the axis s. Similarly, two corresponding tangents t and t' to the conics K and K' intersect in a point P of s.

The dualistic interpretation of Fig. 4 leads to a theorem which has been stated in my dissertation (loc. cit.) and which reads as follows:

Each two conics tangent to each other determine a perspective collineation with the common tangent at their point of tangency as the axis and the intersection-point of their two other common tangents as the center of collineation. (See Fig. 5).

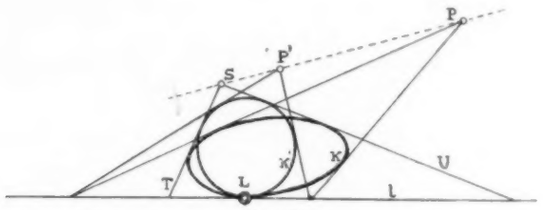


FIG. 5.

The points P and P' represent two collinear points, S the center, and l the axis of perspective collineation.

If, in Fig. 4, we assume K and K' as tangent circles, then the points T and U are the imaginary circular points, and the line s is at infinity. Corresponding lines p and p' are all parallel. This case gives us *similarity of systems in a similar position* and is illustrated in Fig. 6.

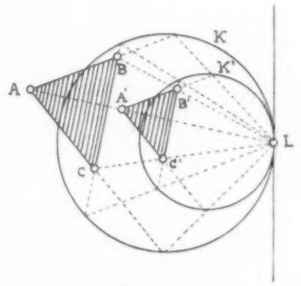


FIG. 6.

We might add a great number of such special cases and, thus, show the usefulness of studying projective transformations in the plane in connection with the congruences of rays (3, 1) and (1, 3). It is, however, easy to develop special cases from our general theorems.

UNIVERSITY OF KANSAS, *July*, 1896.

A GEOMETRIC PICTURE OF THE FIFTEEN SCHOOL GIRL PROBLEM.

By Prof. Ellery W. Davis, Lincoln, Neb.

The problem is to walk out 15 girls by threes, daily for a week, without ever having the same two together.

Instead of girls think of 15 points distributed 8 at the corners of a cube, 6 at the mid-points of the faces, one at the cube's centre.

From them form 35 triads no two having a pair of points in common, thus :—

The centre of the cube with each pair of opposite corners gives 4 triads. Call each triad an a.

The centre of the cube with each pair of mid-points of opposite faces gives 3 triads. Call each a b.

From the 24 triangles, 4 to each face, formed by drawing the face diagonals, select 12. Call each a c.

From the faces of the co-axial inscribed tetrahedron select 4. Call each a d.

Finally, each face-diagonal is a side of a triangle of which the mid-point of the opposite face is a vertex. There are 12 such. Call each an e.

We now easily form the seven sets of 5 triads.

With each a take the perpendicular d and the 3 c that have no points in common with that a or d. This gives 3 of the sets.

With each b take 4 e that have no points in common with it or with each other. This can be done in two ways for the first e that is taken, but the way then chosen leaves no choice for the other 3 e. Thus we have the remaining 4 sets.

It but remains to establish a one-to-one correspondence between the points and the girls.

The system is not 7-cyclic. To prove this notice that if x, y, z is a triad on a face of the cube and o the cube's centre, while x', y', z' are opposite to x, y, z respectively ; then xox', yoy', zoz' and $x'y'z'$ are all triads of the system. Were the system 7-cyclic such a set of 5 triads from 7 elements could not be formed. Try to do so.

The three types of 7-cyclic systems have for Sunday groupings, adopting a common notation,

$A.$ $\quad ka_1b_1\,,\quad a_2a_3a_5\,,\quad a_4b_5b_7\,,\quad a_6b_3b_4\,,\quad a_7b_2b_6\,;$

$B.$ $\quad ka_1b_1\,,\quad a_2a_3a_5\,,\quad a_7b_5b_4\,,\quad a_4b_3b_6\,,\quad a_6b_2b_7\,;$

$C.$ $\quad ka_1b_1\,,\quad a_2a_3a_5\,,\quad a_4b_2b_6\,,\quad a_6b_5b_7\,,\quad a_7b_3b_4\,.$

For o in the $xyzox'y'z'$ set k cannot be taken; for then, aaa for xyz would require bbb for $x'y'z'$, while abb would require baa.

Neither can we have a b for o. This would require with kab, aaa, abb respectively aba, bbb or bbk, baa.

Finally a wont do for o. Here the indices must be considered. The index triads are

$$-, i, i\,; \quad i, i+1, i+3\,; \text{ and } i, i+s, i-s\,;$$

the last form occurring only in C. Then, j is the index of the a chosen for o;

$-, i, i$ would require j, l, l in A or B and $j, l, 2j-i$ in C, the indices belonging to different letters.

$i, i+1, i+3$ would require (in some order) $l, l-1, l-3$ in either A, B, or C.

$i, i+s, i-s$ would require (of course in C) $l, 2j-i-s, 2j-i+s$ with $l \neq 2j-1$.

Each and every requirement being impossible to fulfil the system we have imagined on the cube the system is proved non 7-cyclic.

DEC., 1896.

A SPECIAL FORM OF A QUARTIC SURFACE.

By Prof. J. I. Hutchinson, Ithaca, N. Y.

The quartic surface which is the locus of the vertex of a cone passing through six given points* a, b, c, d, e, f, has for its equation†

$$K = \begin{vmatrix} e_1 f_1 x_2 x_3 x_4 & x_1 & e_1 & f_1 \\ e_2 f_2 x_1 x_3 x_4 & x_2 & e_2 & f_2 \\ e_3 f_3 x_1 x_2 x_4 & x_3 & e_3 & f_3 \\ e_4 f_4 x_1 x_2 x_3 & x_4 & e_4 & f_4 \end{vmatrix} = 0 .$$

The nodal points of the surface are $a, b, \ldots, f$. Their coordinates are respectively, $(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1), (e_1, e_2, e_3, e_4), (f_1, f_2, f_3, f_4)$.

Among the curves lying on the surface, those of especial interest are

(1) the 15 join lines of the nodes ;

(2) the 10 lines, intersections of the 10 pairs of planes determined by the 6 nodes ;

(3) the twisted cubic passing through the nodes.

The coordinates of the surface can be variously expressed as hyperelliptic functions of two variables. For example we may take

$$x_1 : x_2 : x_3 : x_4 = c_{01}\vartheta_{01}\vartheta_3\vartheta_{02}\vartheta_{04} : c_2\vartheta_2\vartheta_1\vartheta_3\vartheta_{01} : c_{03}\vartheta_{03}\vartheta_1\vartheta_{02}\vartheta_{04} : c_4\vartheta_4\vartheta_1\vartheta_3\vartheta_{03} ,$$

$$e_1 : e_2 : e_3 : e_4 = c_{01}c_0c_{23}c_{34} : c_2c_5c_{12}c_{23} : c_{03}c_0c_{12}c_{14} : c_4c_5c_{14}c_{34} ,$$

$$f_1 : f_2 : f_3 : f_4 = c_{01}c_5c_{12}c_{14} : c_2c_0c_{14}c_{34} : c_{03}c_5c_{23}c_{34} : c_4c_0c_{12}c_{23} ,$$

where $\vartheta_\lambda = \vartheta_\lambda(u, v)$, u, and v being two linearly independent hyperelliptic integrals of the first kind, of the form

$$\int \frac{(\alpha z + \beta)\, dz}{\sqrt{a_0 z^6 + a_1 z^5 + \ldots + a_6}}, \text{ and } c_\lambda = \vartheta_\lambda(0, 0) .$$

If now the roots of the sextic $a_0 z^6 + \ldots + a_6$ are in involution it is well known that the hyperelliptic functions of u, v can be expressed in terms of elliptic functions of u and elliptic functions of v by means of a transformation of the second degree.‡

* See Caspary. Bulletin des Sciences Mathématiques, 1891, pg. 308, where a complete list of references is given. Also, Humbert. Journal de Mathématiques, 1893, pg. 463.

† Caspary. Comptes Rendus, t. 112, pg. 1356 (1891).

‡ Cf. my dissertation, On the Reduction of Hyperelliptic Functions ($p = 2$) to Elliptic Functions by a Transformation of the Second Degree. Chicago, 1895.

This leads to the following relations among the theta constants†

$$c_{23}^{\ 2} = -c_{14}^{\ 2}, \quad c_{02}^{\ 2} = c_4^{\ 2}, \quad c_{01}^{\ 2} = c_2^{\ 2}.$$

Bearing these relations in mind we observe that the points e and f depend on one another in such a way that if e be any point (e_1, e_2, e_3, e_4), f must be one of the two points $f' = (e_2, e_1, e_4, e_3)$, $f'' = (e_2, e_1, -e_4, e_3)$.

A geometrical consequence is that the six nodal points lie in involution on the twisted cubic determined by them. To prove this it is only necessary to show that the six points can be joined in pairs by three lines which generate a quadric surface containing the cubic.‡

The equation of the cone $a - bcdef$ is

$$U = \begin{vmatrix} e_2 f_2 x_3 x_4 & e_2 & f_2 \\ e_3 f_3 x_2 x_4 & e_3 & f_3 \\ e_4 f_4 x_2 x_3 & e_4 & f_4 \end{vmatrix} = 0,$$

and of the cone $b - acdef$, is

$$-V = \begin{vmatrix} e_1 f_1 x_3 x_4 & e_1 & f_1 \\ e_3 f_3 x_1 x_4 & e_3 & f_3 \\ e_4 f_4 x_1 x_3 & e_4 & f_4 \end{vmatrix} = 0.$$

These two cones intersect in the line ab and the twisted cubic. The quadric surface

$$Q = e_1 f_1 U + e_2 f_2 V = 0$$

contains the line cd, besides the intersection of U and V.

If now we take for f either one of the points f', or f'', Q is found to contain also the line ef. Its equation reduces to

$$(e_1 e_3 - e_2 e_4)(x_1 x_4 - x_2 x_3) - (e_1 e_4 - e_2 e_3)(x_1 x_3 - x_2 x_4) = 0.$$

Every generator of Q belonging to the same set as ab, cd, ef, meets the cubic in two points which are paired with each other in the involution.

The remaining portion of the intersection of Q with the quartic surface K is a curve of the second order which, it is easy to see, consists of two non-intersecting straight lines. For, the equation of the quartic can be put in the form

$$XY + ZW = 0$$

* Cf. Weber. Crelle, Bd. 84, pg. 353.

† See Reye. Geometrie der Lage, (zweite Auflage), zweiter Abtheilung, pg. 98. Leipzig, 1882.

where

$$X = (e_1e_4 - e_2e_3)(e_2x_3 - e_3x_2)(e_1x_4 - e_4x_1)$$
$$Y = x_1x_4 - x_2x_3$$
$$Z = (e_1e_3 - e_2e_4)(e_2x_4 - e_4x_2)(e_3x_1 - e_1x_3)$$
$$W = x_1x_3 - x_2x_4 .$$

Both Q and K, then, contain the intersection of Y and W,* viz, the lines ab, cd, and

$$(1)\ \begin{cases} x_1 - x_2 = 0 \\ x_3 - x_4 = 0 , \end{cases} \qquad (2)\ \begin{cases} x_1 + x_2 = 0 \\ x_3 + x_4 = 0 . \end{cases}$$

The lines (1) and (2) are new lines on the quartic additional to the 25 lines on the unspecialized surface. As they intersect the lines ab, cd, ef, they belong to a secondary set of generators of Q, and, therefore, each intersects the twisted cubic in one point. The two points d_1 and d_2, so determined are the double-points of the involution. To prove this, it is necessary to show that the primary generators of Q passing through d_1, and d_2 are tangent to the cubic.

The coordinates of d_1 are $(\lambda, \lambda, 1, 1)$ where $\lambda = \dfrac{e_1e_2(e_3 + e_4)}{e_3e_4(e_1 + e_2)}$. The plane

$$x_1 + x_2 - \lambda(x_3 + x_4) = 0$$

meets Q in the line (2), and the primary generator through d_1, which I will call γ_1. Writing Q in the form

$$(e_1e_3 - e_2e_4)[x_3(x_1 + x_2) - x_1(x_3 + x_4)] - (e_1e_4 - e_2e_3)[x_4(x_1 + x_2) - [x_1(x_3 + x_4)] = 0$$

and substituting λ for the ratio $\dfrac{x_1 + x_2}{x_3 + x_4}$ we obtain

$$(e_1 + e_2)(e_3 - e_4)x_1 - \lambda[(e_1e_3 - e_2e_4)x_3 - (e_1e_4 - e_2e_3)x_4] = 0 ,$$

the equation of a plane also containing the line γ_1. But this is the equation of the tangent plane to the cone V at the point d_1. The line γ_1 is, therefore, tangent to the cubic. Similarly for the primary generator of Q through the point d_2.

CORNELL UNIVERSITY, *Aug.*, 1896.

* It is interesting to note that both Y and W meet K in 8 lines each.

CONTENTS.

	Page.
An Analog to DeMoivre's Theorem in a Plane Point System. By Prof. E. W. Hyde,	129
Criteria for Nodes in Dupin's Cyclides, with a Corresponding Classification. By Dr. Virgil Snyder,	137
On the Congruences of Rays (3, 1) and (1, 3). By Dr. Arnold Emch,	148
A Geometric Picture of the Fifteen School Girl Problem. By Prof. Ellery W. Davis,	156
A Special Form of a Quartic Surface. By Prof. J. I. Hutchinson,	158

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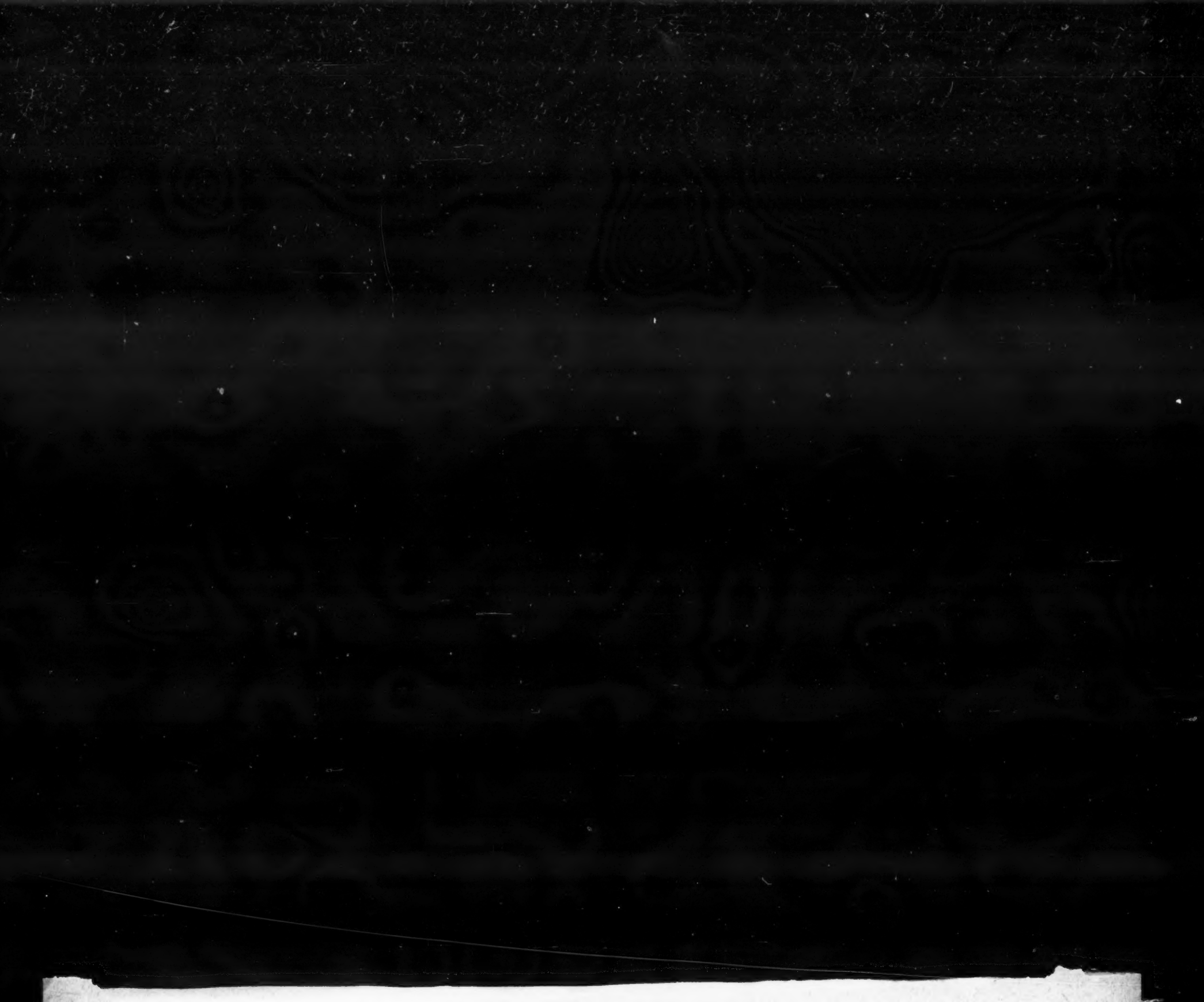
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